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Interacting SUSY-singlet matter in non-relativistic Chern–Simons theory

Yu Nakayama¹, Makoto Sakaguchi² and Kentaroh Yoshida³

¹ Berkeley Center for Theoretical Physics, University of California, Berkeley, CA 94720, USA

² Okayama Institute for Quantum Physics, 1-9-1 Kyoyama, Okayama 700-0015, Japan

³ Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA

E-mail: nakayama@berkeley.edu, makoto.sakaguchi@pref.okayama.jp and kyoshida@kitp.ucsb.edu

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Abstract

We construct an example of supersymmetric Chern–Simons-matter theory with a matter field transforming as a singlet representation of the supersymmetry algebra, where the bosonic and fermionic degrees of freedom do not match. This is obtained as a non-relativistic limit of the $\mathcal{N} = 2$ Chern–Simons-matter theory in 1+2 dimensions, where the particle and anti-particle coexist. We also study the index to investigate the mismatch of bosonic and fermionic degrees of freedom.

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1. Introduction

The Coleman–Mandula theorem [1] and its supersymmetric extension [2] play a significant role in classifying *relativistic* supersymmetry (SUSY) algebras and their representations. The basic dynamical SUSY algebra

$$\{Q, Q^*\} = 2H \quad (1.1)$$

demands that all the relativistic fields should transform as a non-trivial representation of the SUSY algebra. If a field ϕ were a singlet under the SUSY, i.e. $[Q, \phi] = 0$ and $[Q^*, \phi] = 0$, then ϕ would be a singlet under the Hamiltonian time evolution $[H, \phi] = 0$ from the super Jacobi identity. Thus ϕ could not be a dynamical field.

This argument may be modified in the *non-relativistic* (NR) system, where there is no direct analogue of the Coleman–Mandula theorem. Hence there is a theoretical possibility of realizing a theory containing a dynamical field that transforms as a singlet under the SUSY transformation.

Indeed, the Galilean algebra can admit a supersymmetric extension with the so-called kinematical SUSY

$$\{Q, Q^*\} = 2M, \tag{1.2}$$

where M is a mass operator, or more generally an internal symmetry. Then, there is no no-go theorem that forbids a singlet field under the kinematical SUSY charges Q and Q^* . Explicit field theoretical models that possess such a kinematical SUSY algebra alone can be found in the study of the NR Chern–Simons system (e.g. in [3, 4])⁴.

Still, it is quite a non-trivial challenge to construct an example of supersymmetric field theory with a singlet representation under this algebra. One of the reasons is that we do not have a superfield formulation for the NR system. Without the superfield formulation, the SUSY transformation is not independent of the action and both of them should be determined at once, especially when a gauge field is introduced.

In this paper, instead of following the standard manner, we present a theory with a SUSY-singlet matter by taking a NR limit of the relativistic $\mathcal{N} = 2$ Chern–Simons-matter theory [5]. Along the line of discussions in [3], the relativistic Chern–Simons-matter theory is indeed a ubiquitous generating source of many inequivalent (super) Schrödinger invariant field theories. The exotic theory with a SUSY-singlet field, which we have just mentioned and we will pursue here, is also contained in the resulting theories.

The possibility of the SUSY singlet not only stimulates our mathematical curiosity about the representation theory of the NR superconformal algebra, but also has potential physical applications. It is interesting to remark here that the phenomenological prototype of the supersymmetry ($SU(6)$ quark model) first appeared in the context of the non-relativistic hadron theory, where we know ‘singlets’ certainly exist⁵. The existence of singlets is also important in the context of the non-relativistic AdS/CFT correspondence because in the gravity theory, we know that the singlet SUSY transformation is prohibited. It seems curious to understand the possible limiting procedure where this becomes possible. Indeed, we will show how this happens within the field theory.

Since the bosonic and fermionic degrees of freedom do not match in our SUSY-singlet field theories while preserving supersymmetry, it would be interesting to study the index-like object introduced in [10]. We will conclude this paper by computing the index for primary operators of these SUSY-singlet NR Chern–Simons theories.

2. Relativistic $\mathcal{N} = 2$ Chern–Simons-matter theory

The relativistic $\mathcal{N} = 2$ Chern–Simons-matter theory in 1+2 dimensions was originally constructed in [5]. The action is composed of the Chern–Simons term S_{CS} and matter part S_M as follows⁶:

$$S_{\text{rel}} = S_{CS} + S_M,$$

$$S_{CS} = \int dt d^2x \frac{\kappa}{4} \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} = \int dt d^2x \left[\kappa A_0 F_{12} + \frac{\kappa}{2c} \epsilon^{ij} \partial_t A_i A_j \right] \quad (i, j = 1, 2),$$

⁴ Super Schrödinger algebras with dynamical SUSY in 1+2 dimensions are discussed in [5–7]. Super Schrödinger algebras containing only the kinematical SUSY are also presented [6, 7] and the algebras of this type may be related to the gravity dual discussed in [8].

⁵ Historically, the Coleman–Mandula theorem appears as a no-go theorem for relativistic extension of the non-relativistic $SU(6)$ quark models. See [9] for a brief summary of the history and its relevance in the context of the supersymmetry.

⁶ We use the same spinor convention as in [3].

$$S_M = \int dt d^2x \left[-(D_\mu \phi)^* D^\mu \phi - i\bar{\psi} \gamma^\mu D_\mu \psi - \left(\frac{e^2}{\kappa c^2} \right)^2 |\phi|^2 (|\phi|^2 - v^2)^2 + \frac{e^2}{\kappa c^2} (3|\phi|^2 - v^2) i\bar{\psi} \psi \right]. \quad (2.1)$$

The mass m is read off as $m^2 c^2 = \left(\frac{e^2}{\kappa c^2} \right)^2 v^4$. Action (2.1) is invariant under the following SUSY transformation:

$$\begin{aligned} \delta A_\mu &= \frac{e}{\kappa c} \bar{\alpha} \gamma_\mu \psi \phi^* + \frac{e}{\kappa c} \bar{\psi} \gamma_\mu \alpha \phi, \\ \delta \phi &= -i\bar{\alpha} \psi, \\ \delta \psi &= -\gamma^\mu \alpha D_\mu \phi + \frac{e^2}{\kappa c^2} \alpha \phi (v^2 - |\phi|^2). \end{aligned} \quad (2.2)$$

Here α is a two-component complex Grassmann variable and hence (2.1) has $\mathcal{N} = 2$ supersymmetries in 1+2 dimensions.

3. Non-relativistic limit and singlet SUSY

In this section, we study NR limits of the Chern–Simons–matter theory. Since the Chern–Simons part does not show any change in NR limits, the non-trivial difference only appears in the matter sector.

In order to study NR limits of (2.1), first, let us expand the fields as follows:

$$\phi = \frac{1}{\sqrt{2m}} [e^{-imc^2 t} \Phi + e^{imc^2 t} \hat{\Phi}^*], \quad \psi = \sqrt{c} [e^{-imc^2 t} \Psi + e^{imc^2 t} C \hat{\Psi}^*],$$

where $\Psi = (\Psi_1, \Psi_2)^t$ and $\hat{\Psi} = (\hat{\Psi}_1, \hat{\Psi}_2)^t$ are two-component complex Grassmann-valued fields, and $C = i\sigma_2$ is a charge conjugation matrix. Here, ‘hat’ denotes the anti-particle.

The naive NR limit ($c \rightarrow \infty$) by keeping both particle and anti-particle leads to the following matter action⁷:

$$\begin{aligned} S = \int dt d^2x & \left[i\Phi^* D_t \Phi + i\hat{\Phi}^* \hat{D}_t \hat{\Phi} - \frac{1}{2m} [(D_i \Phi)^* D_i \Phi + (\hat{D}_i \hat{\Phi})^* \hat{D}_i \hat{\Phi}] + i\Psi_1^* D_t \Psi_1 \right. \\ & + i\hat{\Psi}_1^* \hat{D}_t \hat{\Psi}_1 - \frac{1}{2m} [(D_i \Psi_1)^* D_i \Psi_1 + (\hat{D}_i \hat{\Psi}_1)^* \hat{D}_i \hat{\Psi}_1] - \frac{e}{2mc} F_{12} (|\Psi_1|^2 - |\hat{\Psi}_1|^2) \\ & \left. + \lambda (|\Phi|^2 + |\hat{\Phi}|^2)^2 + 2\lambda |\Phi|^2 |\hat{\Phi}|^2 + 3\lambda (|\Phi|^2 + |\hat{\Phi}|^2) (|\Psi_1|^2 + |\hat{\Psi}_1|^2) \right], \end{aligned} \quad (3.1)$$

where Ψ_2 and $\hat{\Psi}_2$ have been removed by using the equations of motion,

$$\Psi_2 = -\frac{1}{2mc} D_+ \Psi_1 + \mathcal{O}(1/c^2), \quad \hat{\Psi}_2 = \frac{1}{2mc} \hat{D}_+ \hat{\Psi}_1 + \mathcal{O}(1/c^2). \quad (3.2)$$

We have also introduced the coupling constant $\lambda = \frac{e^2}{2m\kappa}$. It is easy to see that the action is invariant under the bosonic Schrödinger symmetry [11, 12].

The SUSY transformation at the leading order is given by

$$\begin{aligned} \delta_1 \Phi &= -\sqrt{2m\kappa} \alpha^{(1)*} \Psi_1, & \delta_1 \hat{\Phi} &= \sqrt{2m\kappa} \alpha^{(2)} \hat{\Psi}_1, \\ \delta_1 \Psi_1 &= \sqrt{2m\kappa} \alpha^{(1)} \Phi, & \delta_1 \hat{\Psi}_1 &= -\sqrt{2m\kappa} \alpha^{(2)*} \hat{\Phi}, \\ \delta_1 A_0 &= \frac{e}{\sqrt{2m\kappa}} [\alpha^{(1)*} \Psi_1 \Phi^* - \alpha^{(1)} \Psi_1^* \Phi - \alpha^{(2)*} \hat{\Psi}_1^* \hat{\Phi} + \alpha^{(2)} \hat{\Psi}_1 \hat{\Phi}^*], \\ \delta_1 A_i &= 0. \end{aligned} \quad (3.3)$$

⁷ The absolute square of fermions is defined as $|\Psi|^2 = \Psi^* \Psi$.

The SUSY transformation at the next-to-leading order is

$$\begin{aligned}
\delta_2 \Phi &= -\frac{1}{\sqrt{2mc}} \alpha^{(2)*} D_+ \Psi_1, & \delta_2 \hat{\Phi} &= \frac{1}{\sqrt{2mc}} \alpha^{(1)} \hat{D}_+ \hat{\Psi}_1, \\
\delta_2 \Psi_1 &= -\frac{1}{\sqrt{2mc}} \alpha^{(2)} D_- \Phi, & \delta_2 \hat{\Psi}_1 &= \frac{1}{\sqrt{2mc}} \alpha^{(1)*} \hat{D}_- \hat{\Phi}, \\
\delta_2 A_0 &= \frac{e}{(2mc)^{3/2} \kappa} [-\alpha^{(2)*} (D_+ \Psi_1) \Phi^* + \alpha^{(2)} (D_+ \Psi_1)^* \Phi + \alpha^{(1)*} (\hat{D}_+ \hat{\Psi}_1)^* \hat{\Phi} - \alpha^{(1)} (\hat{D}_+ \hat{\Psi}_1) \hat{\Phi}^*], \\
\delta_2 A_+ &= \frac{2ie}{\sqrt{2mck}} [\alpha^{(2)} \Psi_1^* \Phi + \alpha^{(1)*} \hat{\Psi}_1^* \hat{\Phi}], \\
\delta_2 A_- &= \frac{2ie}{\sqrt{2mck}} [\alpha^{(2)*} \Psi_1 \Phi^* + \alpha^{(1)} \hat{\Psi}_1 \hat{\Phi}^*].
\end{aligned} \tag{3.4}$$

These transformations are directly obtained from the NR limit of (2.2). Note that the SUSY parameters $\alpha^{(1)}$ and $\alpha^{(2)}$ are not separated.

The naive NR action $S_{CS} + S$ with (3.1), however, is not invariant even under the SUSY transformation at the leading order. We have no clear understanding of this result, but it is possible to improve the situation by adding a four-Fermi interaction without spoiling any symmetries of the naive action. The improved action is given by

$$\begin{aligned}
S &= \int dt d^2x \left[i\Phi^* D_t \Phi + i\hat{\Phi}^* \hat{D}_t \hat{\Phi} - \frac{1}{2m} [(D_i \Phi)^* D_i \Phi + (\hat{D}_i \hat{\Phi})^* \hat{D}_i \hat{\Phi}] \right. \\
&\quad + i\Psi_1^* D_t \Psi_1 + i\hat{\Psi}_1^* \hat{D}_t \hat{\Psi}_1 - \frac{1}{2m} [(D_i \Psi_1)^* D_i \Psi_1 + (\hat{D}_i \hat{\Psi}_1)^* \hat{D}_i \hat{\Psi}_1] \\
&\quad - \frac{e}{2mc} F_{12} (|\Psi_1|^2 - |\hat{\Psi}_1|^2) + \lambda (|\Phi|^2 + |\hat{\Phi}|^2)^2 + 2\lambda |\Phi|^2 |\hat{\Phi}|^2 \\
&\quad \left. + 3\lambda (|\Phi|^2 + |\hat{\Phi}|^2) (|\Psi_1|^2 + |\hat{\Psi}_1|^2) + 2\lambda |\Psi_1|^2 |\hat{\Psi}_1|^2 \right].
\end{aligned} \tag{3.5}$$

This improved model has four supersymmetries at the leading order and there is no SUSY transformation at the next-to-leading order⁸. This is in accordance with the general expectation from [3]: since the SUSY transformation is not separated from the leading order and the next-to-leading order, the second SUSY is not realized. The improved action now gives a new super Schrödinger invariant field theory with four (real) supercharges. The SUSY charges

$$Q_1 = \sqrt{2m} \int d^2x \Psi_1^* \Phi, \quad Q_2 = -\sqrt{2m} \int d^2x \hat{\Psi}_1^* \hat{\Phi}, \tag{3.6}$$

satisfy the following anti-commutation relations:

$$\{Q_1, Q_1^*\} = 2mN_1, \quad \{Q_2, Q_2^*\} = 2mN_2, \tag{3.7}$$

where we have introduced the particle number density N_1 and the anti-particle number density N_2 defined as, respectively,

$$N_1 = \int d^2x (|\Phi|^2 + |\Psi_1|^2), \quad N_2 = \int d^2x (|\hat{\Phi}|^2 + |\hat{\Psi}_1|^2). \tag{3.8}$$

3.1. Consistent truncation

There are several different ways to take NR limits by reducing the matter contents. We will consider them as follows:

⁸ The next-to-leading SUSY will be resurrected when we only keep particle degrees of freedom as in [5]. See the discussion in the subsequent subsections.

- (i) all particle case: PP ($\hat{\Phi} = \hat{\Psi} = 0$)
- (ii) a singlet SUSY 1: BP ($\hat{\Psi} = 0$)
- (iii) a single SUSY 2: PB ($\hat{\Phi} = 0$).

The sequence of alphabets denotes the degrees of freedom we hold in the NR limit: particle (P), anti-particle (A), both particle and anti-particle (B) and neither of them (N). Since the NR limit preserves the particle number density and the anti-particle number density separately for both bosons and fermions, all the truncations discussed here are consistent (in the sense of strong condition introduced in [3]).

The first case is nothing but [5], so we will not discuss it here except for pointing out the fact that the superconformal symmetry emerges in contrast to the full limit presented above. This is related to the emergence of the dynamical SUSY that is lacking in the full NR action.

Here, we concentrate on more exotic possibilities such as BP or PB to construct a supersymmetric field theory with a singlet representation under the SUSY. It is clear that the other possibilities (up to exchange of particles with anti-particles) PA, BN, NB, PN and NP lead to non-supersymmetric theories. We emphasize the ubiquity of the relativistic Chern–Simons-matter theory to give birth to many inequivalent (super) Schrödinger invariant field theories.

3.2. Singlet SUSY 1

We construct a singlet supersymmetric field theory from the BP case. With our ansatz ($\hat{\Psi} = 0$), the matter action reads

$$\begin{aligned}
 S = \int dt d^2x & \left[i\Phi^* D_t \Phi + i\hat{\Phi}^* \hat{D}_t \hat{\Phi} - \frac{1}{2m} [(D_i \Phi)^* D_i \Phi + (\hat{D}_i \hat{\Phi})^* \hat{D}_i \hat{\Phi}] \right. \\
 & + i\Psi_1^* D_t \Psi_1 - \frac{1}{2m} (D_i \Psi_1)^* D_i \Psi_1 - \frac{e}{2mc} F_{12} |\Psi_1|^2 \\
 & \left. + \lambda(|\Phi|^2 + |\hat{\Phi}|^2)^2 + 2\lambda|\Phi|^2|\hat{\Phi}|^2 + 3\lambda(|\Phi|^2 + |\hat{\Phi}|^2)|\Psi_1|^2 \right]. \tag{3.9}
 \end{aligned}$$

Note that this action (3.9) has no subtlety associated with the four-Fermi term because it vanishes identically due to the ansatz.

The NR action $S_{CS} + S$ with (3.9) is invariant under the following SUSY transformation:

$$\begin{aligned}
 \delta_1 \Phi &= -\sqrt{2mc}\alpha^{(1)*}\Psi_1, & \delta_1 \hat{\Phi} &= 0, \\
 \delta_1 \Psi_1 &= \sqrt{2mc}\alpha^{(1)}\Phi, \\
 \delta_1 A_0 &= \frac{e}{\sqrt{2mck}} [\alpha^{(1)*}\Psi_1\Phi^* - \alpha^{(1)}\Psi_1^*\Phi], \\
 \delta_1 A_i &= 0.
 \end{aligned} \tag{3.10}$$

It is easy to check explicitly that the next-to-leading SUSY transformation (dynamical SUSY) is broken.

It is not difficult to see that the action is invariant under the Schrödinger symmetry, so the NR limit here yields a Schrödinger invariant field theory with two real supercharges. The anti-commutator of the SUSY charges

$$Q = \sqrt{2m} \int d^2x \Psi_1^* \Phi \tag{3.11}$$

can be computed as

$$\{Q, Q^*\} = 2mN_1, \tag{3.12}$$

where N_1 is the conserved charge associated with the number density of particles⁹

$$N_1 = \int d^2x (|\Phi|^2 + |\Psi_1|^2). \quad (3.13)$$

The bosonic field $\hat{\Phi}$ transforms as a singlet under the SUSY transformation, and it does not have its fermionic partner. Nevertheless the field $\hat{\Phi}$ non-trivially interacts with other fields.

It would be possible to consider various supersymmetric deformations of the theory. First, the SUSY does not fix the coefficient in front of $|\hat{\Phi}|^4$, which is a SUSY-singlet, and such a deformation corresponds to a classical marginal deformation of the super Schrödinger invariant theory. Second, we can change the coefficient of

$$\lambda_1 |\Phi|^2 |\hat{\Phi}|^2 + \lambda_2 |\Psi|^2 |\hat{\Phi}|^2$$

as long as we demand

$$\lambda_1 = \lambda_2 + \frac{e^2}{2m\kappa}$$

with electric charge e^2/κ fixed.

3.3. Singlet SUSY 2

Similarly, we can construct a singlet supersymmetric field theory from the PB case. With our ansatz ($\hat{\Phi} = 0$), the matter action reads

$$\begin{aligned} S = \int dt d^2x & \left[i\Phi^* D_t \Phi - \frac{1}{2m} (D_i \Phi)^* D_i \Phi + i\Psi_1^* D_t \Psi_1 + i\hat{\Psi}_1^* \hat{D}_t \hat{\Psi}_1 \right. \\ & - \frac{1}{2m} [(D_i \Psi_1)^* D_i \Psi_1 + (\hat{D}_i \hat{\Psi}_1)^* \hat{D}_i \hat{\Psi}_1] - \frac{e}{2mc} F_{12} (|\Psi_1|^2 - |\hat{\Psi}_1|^2) \\ & \left. + \lambda |\Phi|^4 + 3\lambda |\Phi|^2 (|\Psi_1|^2 + |\hat{\Psi}_1|^2) + 2\lambda |\Psi_1|^2 |\hat{\Psi}_1|^2 \right]. \end{aligned} \quad (3.14)$$

Here, we have used improved action and added the four-Fermi term.

The NR action $S_{CS} + S$ with (3.14) is invariant under the following SUSY transformation:

$$\begin{aligned} \delta_1 \Phi &= -\sqrt{2m\kappa} \alpha^{(1)*} \Psi_1, \\ \delta_1 \Psi_1 &= \sqrt{2m\kappa} \alpha^{(1)} \Phi, \quad \delta_1 \hat{\Psi}_1 = 0, \\ \delta_1 A_0 &= \frac{e}{\sqrt{2m\kappa}} [\alpha^{(1)*} \Psi_1 \Phi^* - \alpha^{(1)} \Psi_1^* \Phi], \\ \delta_1 A_i &= 0. \end{aligned} \quad (3.15)$$

It is an easy task to check that the next-to-leading SUSY transformation (dynamical SUSY) is broken.

It is not difficult to see that the action is invariant under the Schrödinger symmetry, so the NR limit here yields a Schrödinger invariant field theory with two real supercharges. The anti-commutator of the SUSY charges

$$Q = \sqrt{2m} \int d^2x \Psi_1^* \Phi \quad (3.16)$$

can be computed as

$$\{Q, Q^*\} = 2mN_1, \quad (3.17)$$

⁹ In addition, the model has two $U(1)$ symmetries associated with the total mass operator and the fermion number.

where N_1 is the conserved charge associated with the number density of particles¹⁰

$$N_1 = \int d^2x (|\Phi|^2 + |\Psi_1|^2). \quad (3.18)$$

The fermionic field $\hat{\Psi}_1$ transforms as a singlet under the SUSY transformation, and it does not have its bosonic partner. Nevertheless the field $\hat{\Psi}_1$ non-trivially interacts with other fields.

The model is essentially obtained by replacing $\hat{\Phi}$ by $\hat{\Psi}_1$ in the BP theory because $\hat{\Phi}$ is a singlet representation and it can simply be replaced with another singlet field $\hat{\Psi}_1$. The additional Pauli interaction, which does not exist in the BP theory, is also a SUSY-singlet, so there is no problem here.

More generally, if a SUSY-singlet piece $F(Q_i)$ could be constructed out of fields Q_i with non-trivial representations of SUSY, it would be possible to couple it to a SUSY-singlet field S in a SUSY invariant Lagrangian as

$$\delta L = f(S, \partial S) \times F(Q_i).$$

As is discussed in the introduction, for dynamical SUSY (as in the relativistic case), there is no such a candidate of $F(Q_i)$. This is equivalent to the well-known fact that the relativistic supersymmetric field theory does not have a SUSY invariant Lagrangian, but the invariance is always only up to total derivative terms.

The kinematical SUSY allows nontrivial $F(Q_i)$ as we have explicitly seen in this section. The form of $f(S, \partial S)$ (as well as $F(Q_i)$) is partially fixed by the classical Schrödinger invariance. It would be interesting to see whether the quantum Schrödinger invariance makes the parameters of the theory more restrictive¹¹.

4. Index for primary operators

Our BP theory and PB theory do not have balanced bosonic degrees of freedom and fermionic degrees of freedom. Because of this mismatch, we may suspect that the virtue of SUSY (namely, the Bose–Fermi cancellation) might be lost. From the anti-commutation relation, however, we can see that the only mismatch of the degrees of freedom and the consequent lack of Bose–Fermi cancellation could appear only in the zero N_1 sector. In addition, the nonzero N_1 sector may be taken as a superselection sector due to the particle number conservation in the NR system. With this regard, we would like to study the index-like object $\text{Tr}(-1)^F e^{-\beta N_1}$ in this section.

The NR supersymmetric conformal field theories discussed in the previous section have a non-trivial involutive anti-automorphism (see [10] for details). We can use it to define the index as

$$I(x) = \text{Tr}(-1)^F e^{-\beta N_1} x^D, \quad (4.1)$$

where $2mN_1 = \{Q, Q^*\}$ for our BP, PB theories. This index counts the operators annihilated by Q^* , and we will see that the index does not depend on β . In order to distinguish operators that contribute to the index, we have introduced the chemical potential x that couples to the dilatation D . Note that D commutes with Q and Q^* so that the Bose–Fermi cancellation is intact.

Because of the complicated singular-vector structure of the zero particle number ($M = 0$) sector in the representation theory of a Schrödinger group, we only study the index for primary

¹⁰ In addition, the model has two $U(1)$ symmetries associated with the total mass operator and the fermion number.

¹¹ For example, the $\mathcal{N} = 2$ PP limit [5] has vanishing beta functions. See [3] and references therein.

Table 1. List of the letters contributing to the index for primary operators.

Letters	Φ	Φ^*	$\hat{\Phi}$	$\hat{\Phi}^*$	Ψ_1	Ψ_1^*	$\hat{\Psi}_1$	$\hat{\Psi}_1^*$
N_1	+1	-1	0	0	+1	-1	0	0
M	+1	-1	+1	-1	+1	-1	+1	-1
D	1	1	1	1	1	1	1	1
$f(\theta, \beta, x)$	$x e^{-\beta} e^{i\theta}$	$x e^{\beta} e^{-i\theta}$	$x e^{i\theta}$	$x e^{-i\theta}$	$-x e^{-\beta} e^{i\theta}$	$-x e^{\beta} e^{-i\theta}$	$-x e^{i\theta}$	$-x e^{-i\theta}$

operators annihilated both by the Galilean boost G_i and the special conformal transformation K . In the $e \rightarrow 0$ limit, the computation of the index boils down to the counting of gauge invariant operators, which may be obtained by the integration over the $U(1)$ holonomy [10] as

$$I_p(x) \equiv \text{Tr}_{\text{primary}} (-1)^F e^{-\beta N_1} x^D = \int_0^{2\pi} \frac{d\theta}{2\pi} \exp \left[\sum_i \sum_{n=1}^{\infty} \frac{1}{n} f_i(n\theta, n\beta, x^n) \right], \quad (4.2)$$

where the summation i is over the species of primary fields and the corresponding single particle indices f_i are shown in table 1. The direct integration gives the index for primary operators as

$$I_p(x; \text{BP}) = \frac{1}{1-x^2}, \quad I_p(x; \text{PB}) = 1-x^2. \quad (4.3)$$

The index is independent of β due to the Bose–Fermi cancellation in the nonzero N_1 sector as advocated, but the nonzero index shows a mismatch between the bosonic and fermionic degrees of freedom in the zero N_1 sector.

5. Outlook and conclusion

In this paper, we have studied the singlet representation of the supersymmetry in the context of the non-relativistic Chern–Simons–matter theory. While there is no no-go theorem for such a singlet representation in a non-relativistic system, it is not obvious how to construct invariant actions. We have obtained the explicit action from the careful non-relativistic limit of the relativistic Chern–Simons–matter theory.

Once we have the singlet representation of SUSY, the holy grail of the SUSY, i.e. the Bose–Fermi cancellation, is lost. We have shown that this only occurs in the zero-particle number sector, and we have studied the contribution to the index. The lack of the Bose–Fermi cancellation may lead to possible renormalization running of the coupling constants of the Chern–Simons matter theory considered in this paper. With this regard, we have noted that our theory allows classical marginal deformations, and it would be interesting to study the beta function and renormalization group flows.

Another direction to pursue is possible generalization to higher dimensions. If we could throw away the gauge interaction and restrict ourselves to the case with only potential terms, it would not be so difficult to obtain the similar singlet representation of the non-relativistic SUSY theories in higher dimensions. A real challenge, however, is how to introduce the gauge field: (1+2) dimension is special in the sense that the Chern–Simons action is topological and in particular Galilean invariant, but a simple, or the most relevant gauge invariant action does not allow Galilean invariance in higher dimensions. We leave it as an interesting future problem to be studied.

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